PROJET DE FIN D'ETUDES

pour obtenir le diplôme de

Université Galatasaray
Spécialité : **Mathématiques**
Directeur : **Meral TOSUN**

UNE BRÈVE INTRODUCTION À LA GÉOMÉTRIE TROPICAL

préparée par **ZEYNEP TOPCU**

Juin '11

THÈSE

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Chapter 1

Introduction

We want to study the *solution* set $V(f_1, \ldots, f_s)$ of a system of equations

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ f_s(x_1, \ldots, x_n) &= 0 \end{aligned}$$

where the f_i are polynomials.

The first case to consider would be $n = s = 1$, that is, we have a single polynomial

$$f = a_m \cdot x^m + a_{m-1} \cdot x^{m-1} + \ldots + a_1 \cdot x + a_0$$

and we are looking for the *roots* of that polynomial. In theory, it is not too hard to deal with. There are at most n roots, and if the base field is algebraically closed like $\mathbb{C}$, then there are exactly n roots counted with the appropriate multiplicity. However, it is very hard in practice to find any of the roots.
The second case to consider would have n and s arbitrary, but the polynomials are all linear, *i.e.* they are of the form

$$f = a_{i1} \cdot x_1 + \ldots + a_{in} \cdot x_n + b_i.$$

Then the solution set is an $afine\ vector\ space$.

Tropical geometry is rather a new area of mathematics. The name "Tropical" comes from the country of origin of the Brazilian mathematician Imre Simon. It is similar to Algebraic Geometry. To be more precise let us see some tools in both area:
In algebraic geometry we consider polynomials over the fields where $+$

and $\cdot$ are the algebraic operations. This means that all the coefficients and the exponents of the polynomial are in a field. The common zeros of a finite number of polynomials defines an affine variety.

In tropical geometry, instead of working over a field we work over a semiring in which $\bigoplus$ and $\bigodot$ are tropical operations. In fact we first consider a polynomial defined as in algebraic geometry then we tropicalize it and we get a tropical polynomial defined in a semiring.

In Chapter 3, we introduce tropicalization of polynomials. We are especially interested in the hypersurfaces which means $s = 1$ above. We examine tropicalization of polynomials and consider the non-linear locus of the tropical polynomials. Which gives us polyhedral complex.

Acknowledgements

First of all i would like to thank Meral Tosun for bracing me up, supporting and believing me from beginning of the this study. She has brightened me with her profound knowledge and directed me during this study.

Also i would like to thank my professors Muhammed Uludag, Aysegül Ulus and Gülay Kaya in helping me to broaden my view and knowledge.

Also i would like to deeply thank my family and friends for their supports during this study.

Chapter 2

Preliminaries

A semiring is an algebraic structure consisting of a set together with two binary operations addition and multiplication.A set S is called a semiring if it satisfies the conditions:
(i) $\mathbb{S}$ is closed, associative and commutative under addition and multiplication,
$(ii$ Multiplication is distributive over the addition,
(iii) There exists neutral element for both operations addition and multiplication.

Example 1 *Let us show that $\mathbb{Z}^+$ is a semiring. Let $a, b, c \in \mathbb{Z}$.*
(i) *Closure:*

$$a + b \in \mathbb{Z}$$
$$a.b \in \mathbb{Z}$$

(ii) *Associativity:*

$$(a + b) + c = a + (b + c)$$
$$(a.b).c = a.(b.c)$$

(iii) *Commutativity:*

$$a + b = b + a$$
$$a.b = b.a$$

(iv) *Neutral Element:*

$$a + 0 = 0 + a = a$$
$$a.1 = 1.a = a$$

(v) *Distrubutivity:*

$$a.(b + c) = (a.b) + (a.c)$$
$$(a + b).c = (a.c) + (b.c)$$

(vi) *Additive Inverse:*

For all positive integer a, there exists $b \in \mathbb{Z}^+$ such that $a + b = b + a = o$.
By the equation we get $b = -a$. This contradicts $\mathbb{Z}^+$

If, in addition a semiring satisfies the requirement that each element must have an additive inverse it is called a *ring*.

Let us see that $\mathbb{R}$ is a ring. Let $a, b, c \in \mathbb{R}$.

(i) Closure:

$$a + b \in \mathbb{R}$$
$$a.b \in \mathbb{R}$$

(ii) Associativity:

$$(a + b) + c = a + (b + c)$$
$$(a.b).c = a.(b.c)$$

(iii) Commutativity:

$$a + b = b + a$$
$$a.b = b.a$$

(iv) Neutral Element:

$$a + 0 = 0 + a = a$$
$$a.1 = 1.a = a$$

(v) Distributivity:

$$a.(b + c) = (a.b) + (a.c)$$
$$(a + b).c = (a.c) + (b.c)$$

(vi) Additive Inverse:

$$\forall a \in \mathbb{R}, \exists b \in \mathbb{R} \text{ s.t. } a + b = b + a = o$$

2.1 Algebraic Geometry

Definition 2.1.1 *A monomial in* $x_1, x_2, \ldots, x_n$ *is a product of the form*

$$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

where the exponents $\alpha_1, \ldots, \alpha_n$ *are all nonnegative integers.*

Definition 2.1.2 *Consider the polynomial ring* $k[x_1, \ldots, x_n]$*. Let* $f_1, f_2, \ldots, f_s$ *be polynomials in* $k[x_1, \ldots x_n]$*. An ideal* $I =< f_1, \ldots, f_s >$ *in* $k[x_1, \ldots, x_n]$ *is of the form*

$$I =< f_1, \ldots, f_s >= \{\sum_{i=1}^{s} h_i f_i \,|\, h_1, \ldots, h_s \in k[x_1, \ldots, x_n]\}$$

The polynomials $f_1, \ldots, f_s$ *are called generators of the ideal.*

Definition 2.1.3 *Let* k *be a field and let* $f_1, \ldots, f_s$ *be polynomials in* $k[x_1, \ldots, x_n]$*. Then we set*

$$V(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in k^n | f_i(a_1, \ldots, a_n) = 0 \, for \; all \, 1 \leq i \leq s\}$$

.

$V(f_1, \ldots, f_s)$ *is called affine variety defined by* $f_1, \ldots, f_s$*.*

If $s = 1$*,* $V(f)$*is called an hypersurface.*

Chapter 3

Introduction to Tropical Geometry

Let $\mathbb{S}$ be a semiring with the operations $\bigoplus$ and $\bigodot$ defined as

$$x \bigoplus \text{y} = \min(x,\text{y})$$

$$x \bigodot \text{y} = x+\text{y}$$

Example 2 *a)* $3 \bigoplus 5 = min(3,5) = 3$

b) $3 \bigodot 5 = 3 + 5 = 8$

The operations $\bigoplus$ and $\bigodot$ are called tropical addition and tropical multiplication respectively. We have the following tables:

$\bigoplus$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

$\bigodot$	1	2	3	4	5
1	2	3	4	5	6
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10

Example 3 *Let us see that* $(\mathbb{R} \cup \{\infty\}, \bigoplus, \bigodot)$ *is a tropical semiring.*

We need to show that $\mathbb{R} \cup \{\infty\}$ *satisfies the properties of a semiring under tropical operations* $\bigodot$ *and* $\bigoplus$*:*

Let $x, y, z \in \mathbb{R} \cup \{\infty\}$

(i) *Associativity:*

$$x \bigoplus (y \bigoplus z) = min(x, min(y, z)) = min(x, y) \ or \ min(x, z)$$

$$= min(x, y, z)$$

$$= min(min(x, y), z)$$

$$= (x \bigoplus y) \bigoplus z$$

$$\implies x \bigoplus (y \bigoplus z) = (x \bigoplus y) \bigoplus z$$

$$x \bigodot (y \bigodot z) = x + (y + z)$$

$$= (x + y) + z \ by \ the \ associativity \ of \ (\mathbb{R}^n, +, \cdot)$$

$$= (x \bigodot y) \bigodot z$$

$$\implies x \bigodot (y \bigodot z) = (x \bigodot y) \bigodot z$$

(ii) *Commutativity:*

$$x \bigoplus y = min(x, y)$$

$$= min(y, x)$$

$$= y \bigoplus x$$

$$\implies x \bigoplus y = y \bigoplus x$$

$$x \bigodot y = x + y$$

$$= y + x \ by \ the \ additivity \ in \ (\mathbb{R}^n, +, \cdot)$$

$$= y \bigodot x$$

$$\implies x \bigodot y = y \bigodot x$$

(iii) Distributivity:

$$x \bigodot (y \bigoplus z) = x + min(y, z)$$

$$= min(x + y, x + z)$$

$$= (x \bigodot y) \bigoplus (x \bigodot z)$$

$$\Longrightarrow x \bigodot (y \bigoplus z) = (x \bigodot y) \bigoplus (x \bigodot z)$$

(iv) Neutral element:

Let y be neutral element for tropical addition. Then tropical addition of y and arbitrary real number gives us the following equation:

$$x \bigoplus y = min(x, y) = x$$

This is true for all x in $\mathbb{R} \cup \{\infty\}$. Since x can be chosen sufficiently big, y must be equal to ∞.

Let e be neutral element for tropical multiplication. Then we have:

$$x \bigodot e = x + e$$

Tropical multiplication behaves as our usual addition. Therefore neutral element e is the zero element of $\mathbb{R}$.

Remark 3.0.4 *In usual algebraic operations, the division operation is the inverse to the multiplication. But the tropical division is defined:*

$$x \oslash y = x - y$$

Example 4 $8 \oslash 5 = 8 - 5 = 3$

Remark 3.0.5 *There is no subtraction operation under tropical semirings:*
Let us consider x as 5 minus 2. This will give the equation $2 \bigoplus x = 5$ which is impossible under tropical addition defined above.

Matrix Multiplication

Let $(u_1, u_2, \ldots.., u_n)$ and $(v_1, v_2, \ldots., v_n)$ be two vectors in $\mathbb{R}^n \cup \{\infty\}$. We calculate tropical multiplication of a row vector and a column vector as follows

$$\begin{aligned}(u_1, u_2, \ldots.., u_n) \bigodot (v_1, v_1, \ldots., v_n)^T &= (u_1 \bigodot v_1 \bigoplus u_2 \bigodot v_2 \bigoplus \ldots.. \bigoplus u_n \bigodot v_n)\\ &= (u_1 + v_1 \bigoplus u_2 + v_2 \bigoplus \ldots u_n + v_n)\\ &= min(u_1 + v_1, \ldots, u_n + v_n)\end{aligned}$$

Example 5 *Let $(2, -1, 3)$ and $(-2, -3, 5)^T$ be a row and column vectors. Then tropical multiplication of these two vectors is*

$$\begin{aligned}(2, -1, 3) \bigodot (-2, -3, 5)^T &= (2 \bigodot -2 \bigoplus -1 \bigodot -3 \bigoplus 3 \bigodot 5)\\ &= (0 \bigoplus -4 \bigoplus 8)\\ &= max(0, -4, 8) = -4\end{aligned}$$

Example 6 *Let us examine tropical multiplication of two matrices of size* 2 *x* 2.

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \bigodot \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} min(1+2, 0+4) & min(1+1, 0+3) \\ min(2+2, 3+4) & min(2+1, 3+3) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \bigodot \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

Tropical Factorization

Example 7 *Let f be a polynomial such that*

$$f(x) = x^2 \bigoplus 1 \bigodot x \bigoplus 4 = min(2x, x+1, 4)$$

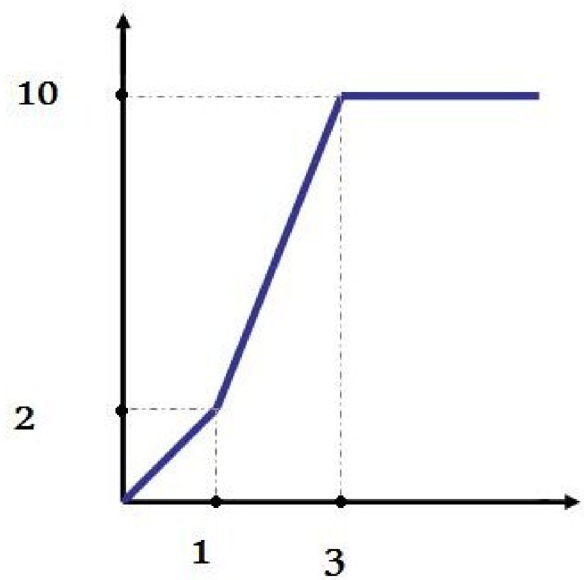
. Deduced factorization of $f(x)$:

$$(x \bigoplus 1) \bigodot (x \bigoplus 3) = (x \bigodot x) \bigoplus (x \bigodot 3) \bigoplus (1 \bigodot x) \bigoplus (1 \bigodot 3)$$

$$= 2x \bigoplus x+3 \bigoplus x+1 \bigoplus 4$$

Thus $f(x) = min(2x, x+1, 4)$.

We obtain the same solution by two different polynomials.



3.1 Tropicalization

Definition 3.1.1 *Let $x = (x_1, \dots, x_n)$ be a variable in $\mathbb{R}^n$ and let $\alpha_1, \dots, \alpha_n$ be variables in the ring $k[x_1, \dots, x_n]$.*
A tropical monomial is

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

It is also called the tropicalization of the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

Example 8 *Let us consider monomial $x_1^3 x_2 x_3^2$. Tropicalization of this monomial is the linear function*

$$\begin{aligned} x_1^3 x_2 x_3^2 &= x_1 \bigodot x_1 \bigodot x_1 \bigodot x_2 \bigodot x_3 \bigodot x_3 \\ &= x_1 + x_1 + x_1 + x_2 + x_3 + x_3 \\ &= 3x_1 + x_2 + 2x_3 \end{aligned}$$

Remark 3.1.2 *A tropical monomial represents a function from $\mathbb{R}^n$ to $\mathbb{R}$ which sends $(x_1^{\alpha_1} \ldots x_n^{\alpha_n})$ to $\alpha_1 x_1 + \ldots + \alpha_n x_n$.*

Let $trop(f) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a polynomial. $trop(f)$ is defined as a linear combination of tropical monomials, which is of the form,

$$trop(f) = \bigoplus_{(i,j)} c_{ij} \bigodot x^i \bigodot y^j$$

where c_{ij}'s are the real numbers.

Let $a_1, a_2, \ldots$ be real numbers and let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}$ be variables in $k[x_1, \ldots, x_n]$.
Consider polynomial $f(x_1, \ldots, x_n) : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$f(x_1, \ldots, x_n) = a_1 x_1^{\alpha_1^{(1)}} x_2^{\alpha_2^{(1)}} \ldots x_n^{\alpha_n^{(1)}} + a_2 x_1^{\alpha_1^{(2)}} x_2^{\alpha_2^{(2)}} \ldots x_n^{\alpha_n^{(2)}} + \ldots + a_n x_1^{\alpha_1^{(n)}} x_2^{\alpha_2^{(n)}} \ldots x_n^{\alpha_n^{(n)}}$$

Then tropicalization of this polynomial is

$$trop(f) = min(a_1 + \alpha_1^{(1)} x_1 + \ldots + \alpha_n^{(1)} x_n, \ldots, a_n + \alpha_1^{(n)} x_1 + \ldots + \alpha_n^{(n)} x_n).$$

Theorem 3.1.3 *Let $trop(f) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial. Then $trop(f)$ satisfies three properties:*
(i) $trop(f)$ is continuous
(ii) $trop(f)$ is piecewise-linear
(iii) $trop(f)$ is concave

Example 9 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a polynomial. For all $x \in \mathbb{R}$, $f(x) = 3x^2 + 2x^3$. Then tropicalization of $f(x)$ is:*

$$trop(f) = min(2x + 3, 3x + 2)$$

Example 10 *Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a polynomial with $g(x, y) = 4x^2 + 2y^3 + 3$. The tropicalization of $g(x, y)$ is*

$$trop(g) = min(2x + 4, 3y + 2, 3)$$

Remark 3.1.4 *Tropical addition can be also defined as the maximum of two variables, as follows:*

$$x \bigoplus y = max(x, y)$$

Example 11 $8 \bigoplus 3 = max(8, 3) = 8$

Definition 3.1.5 *Let $trop(f) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial. The hypersurface defined by $trop(f)$ is the set of all points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ at which the minimum is attained at least twice.*

What are the roots of $trop(f)$?

Let us consider the roots of $x^3 \bigoplus 2x^2 \bigoplus 6x \bigoplus 11$. Then, $trop(f) = min(3x, 2x + 2, x + 6, 11)$.
We have three inequalities:

$$(i)\, 3x = 2x + 2 \leq x + 6 \leq 11. \text{ This gives us } x = 2.$$
$$(ii)\, x + 6 = 2x + 2 \leq 11 \leq 3x. \text{ By this inequality we get } x = 4.$$
$$(iii)\, x + 6 = 11 \leq 2x + 2 \leq 3x. \text{ Then we have } x = 5$$

Then the roots of $trop(f)$: $x = 2$, $x = -4$ adn $x = 5$. According to roots we get:

for $x = 2$ $\quad trop(f) = min(\underline{6}, \underline{6}, 8, 11)$

for $x = 4$ $\quad trop(f) = min(12, \underline{10}, \underline{10}, 11)$.

for $x = 5$ $\quad trop(f) = min(15, 12, \underline{11}, \underline{11})$.
The minimum is attained at least twice.

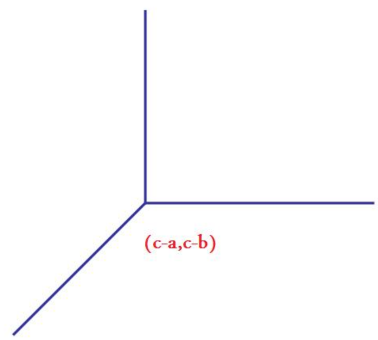
3.2 Graph of $trop(f)$

Given a tropical polynomial $f(x,y)$ its curve $\mathbb{T}(f)$ is the set of points $(x,y) \in \mathbb{R}^2$ where the minimum is attained twice. The graph of $trop(f)$ includes all points $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ with the property that f is not linear at x.

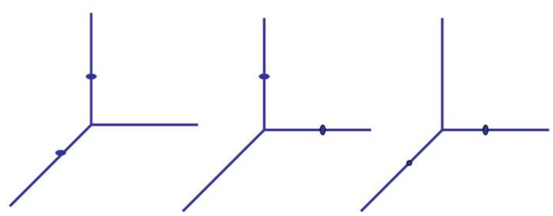
Proposition 3.2.1 *$x \in \mathbb{R}$ lies in graph of $trop(f)$ if and only if $trop(f)$ is not linear at x.*

Lines: $f(x,y) = a \bigodot x \bigoplus b \bigodot y \bigoplus c$

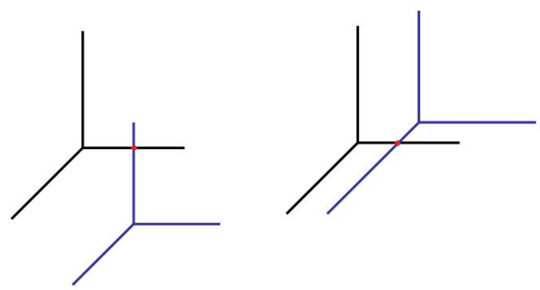
$$= max(a + x, b + y, c)$$



Fact 1 Any two points span a unique line.



Fact 2 Any two lines meet in a unique point.



Examples of Tropical Lines To draw graph of a tropical polynomial, we can take "min" as well as "max". In our first example belove we treat these two cases:

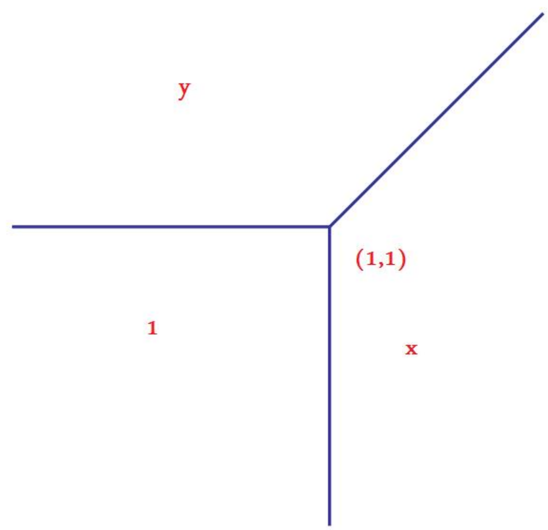
Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a polynomial.

$$(x, y) \longmapsto x + y + 1$$

Tropicalization of this polynomial will be

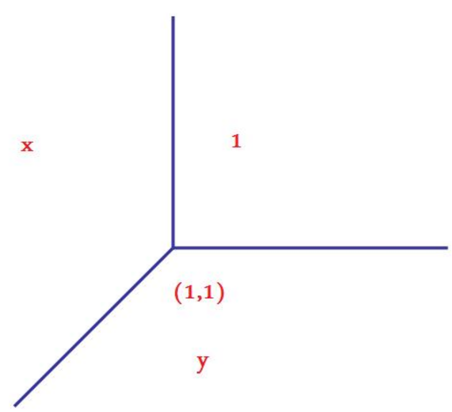
$(i) trop(f) = max(x, y, 1)$.
We look at the points where $trop(f)$ is piecewise-linear. We have three inequalities:

(1) $x = y \geq 1$ This gives the locus where $x = y$, $x \geq 1$ and $y \geq 1$
(2) $y = 1 \geq x$ This gives the locus where $y = 1$, $y \geq x$ and $1 \geq x$
(3) $x = 1 \geq y$ This gives the locus where $x = 1$, $x \geq y$ and $1 \geq y$
The intersection of the lines $x = y$,$x = 1$ and $y = 1$ is the point $(1, 1)$ at which $trop(f)$ is not locally linear.



$(ii) trop(f) = min(x, y, 1)$.
We look at the points where $trop(f)$ is piecewise-linear. We have three inequalities:

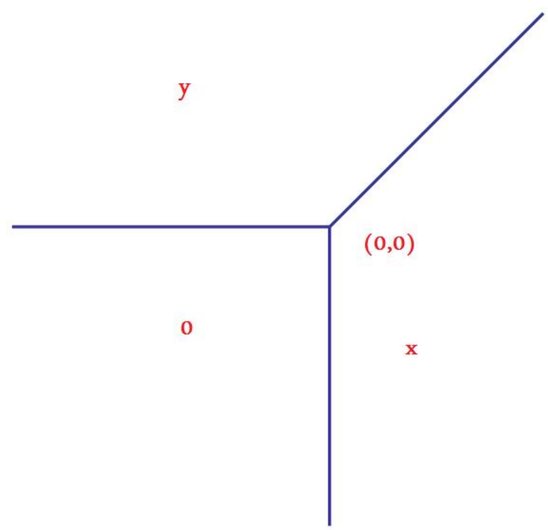
(1) $x = y \leq 1$ This gives the locus where $x = y$, $x \leq 1$ and $y \leq 1$
(2) $y = 1 \leq x$ This gives the locus where $y = 1$, $y \leq x$ and $1 \leq x$
(3) $x = 1 \leq y$ This gives the locus where $x = 1$, $x \leq y$ and $1 \leq y$
The intersection of the lines $x = y$,$x = 1$ and $y = 1$ is the point $(1, 1)$ at which $trop(f)$ is not locally linear.



$(iii) trop(f) = max(x, y, 0)$.
We look at the points where $trop(f)$ is piecewise-linear. We have three

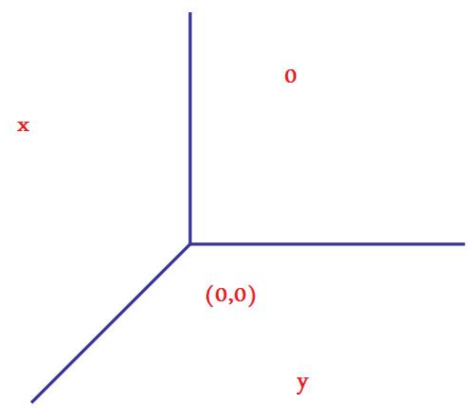
inequalities:

(1) $x = y \geq 0$ This gives the locus where $x = y$, $x \geq 0$ and $y \geq 0$
(2) $y = 0 \geq x$ This gives the locus where $y = 0$, $y \geq x$ and $0 \geq x$
(3) $x = 0 \geq y$ This gives the locus where $x = 0$, $x \geq y$ and $0 \geq y$
The intersection of the lines $x = y$,0 and $y = 0$ is the point $(0, 0)$ at which $trop(f)$ is not locally linear.



$(iv) trop(f) = min(x, y, 0)$.
We look at the points where $trop(f)$ is piecewise-linear. We have three inequalities:

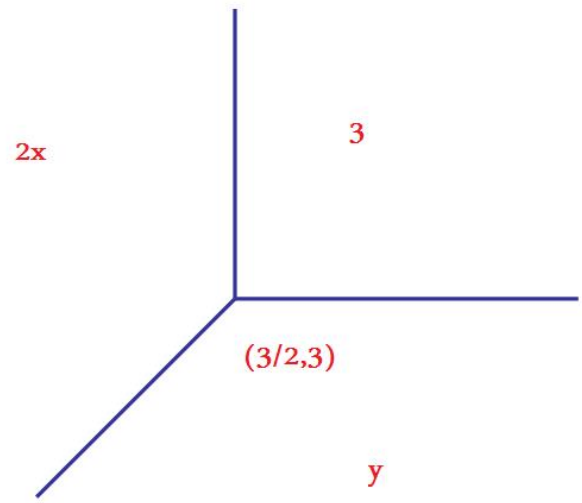
(1) $x = y \leq 0$ This gives the locus where $x = y$, $x \leq 0$ and $y \leq 0$
(2) $y = 0 \leq x$ This gives the locus where $y = 0$, $y \leq x$ and $0 \leq x$
(3) $x = o \leq y$ This gives the locus where $x = 0$, $x \leq y$ and $0 \leq y$
The intersection of the lines $x = y$,$x = 0$ and $y = 0$ is the point $(0, 0)$ at which $trop(f)$ is not locally linear.



Example 12 *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a polynomial.*
$$(x, y) \longmapsto x^2 + y + 3$$
$trop(f) = min(2x, y, 3)$ *is the tropicalization of this polynomial. We have:*

(1) $2x = y \leq 3$ *This gives the locus where* $2x = y$, $2x \leq 3$ *and* $y \leq 3$
(2) $2x = 3 \leq y$ *This gives the locus where* $2x = 3$, $2x \leq y$ *and* $3 \leq y$
(3) $y = 3 \leq 2x$ *This gives the locus where* $y = 3$, $y \leq 2x$ *and* $3 \leq 2x$

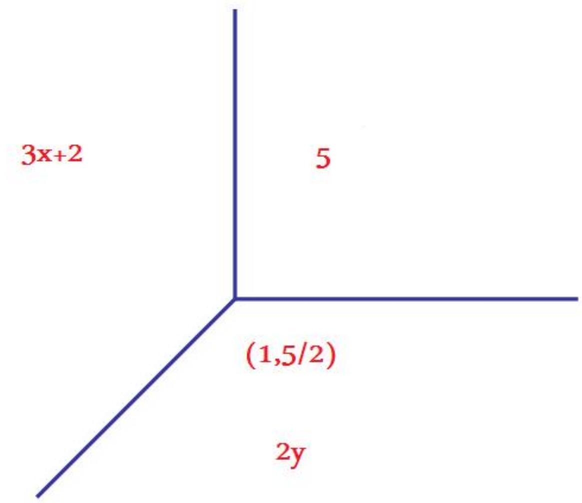
The intersection of the lines $2x = y$, $x = 3/2$ *and* $y = 3$ *is the point* $(3/2, 3)$.



Example 13 *Let us see tropical hypersurfaces of polynomial* $f(x, y) = 2x^3 + y^2 + 5$.
Tropicalization of this polynomial is

$trop(f) = min(3x + 2, 2y, 5)$. *Then we have three inequalities:*

(1) $3x + 2 = 2y \leq 5$
(2) $3x + 2 = 5 \leq 2y$
(3) $2y = 5 \leq 3x + 2$.
By $(1), (2)$ *and* (3) *we can find the point* $x = (1, 5/2)$ *in* $\mathbb{R}^2$ *where the function is not locally linear. Then with respect to inequalities we draw lines of the function.*

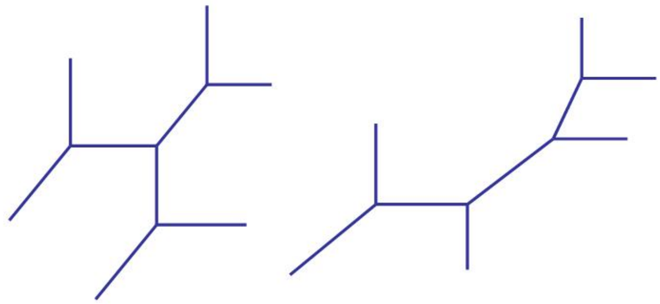


Examples of Tropical Curve

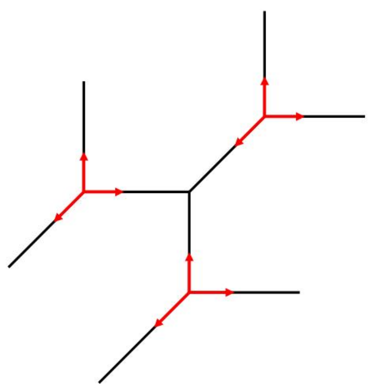
Example 14 *Let us consider quadratic function* $f(x, y) = ax^2 + bx + cxy + dy^2 + ey + f$. *The tropicalization of this function gives*

$$trop(f) = max(2x + a, x + b, x + y + c, 2y + d, y + e, f).$$

(1) $2x + a = x + y + c$
(2) $2x + a = 2y + d$
(3) $2x + a = y + e$
(4) $2x + a = f$
(5) $x + b = 2y + d$
(6) $x + b = y + e$*....etc.*



Remark 3.2.2 *The graph of a quadratic curve has two lines in each of directions.*

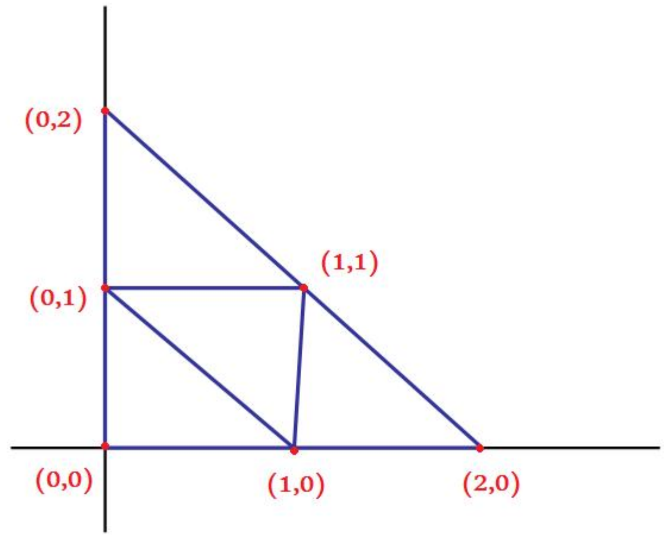


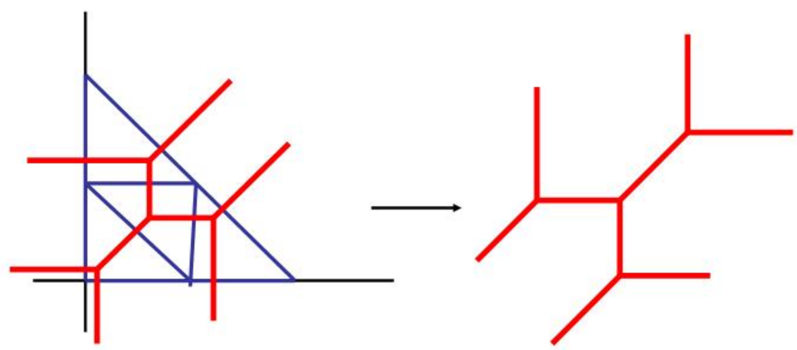
3.3 Newton Triangle

Remark 3.3.1 *There exist another method to see the graph of a tropical polynomial. The Newton polygon of a polynomial $f(x, y)$ is the convex hull of half points (i, j) such that $x^i y^j$ appears in $f(x, y)$.*

Example 15 *Let $d = 2$, $f = tx^2 + xy + ty^2 + x + y + t^6$.*

$$A_1 = \{(2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)\}$$


(0,2)
(1,1)
(0,1)
(0,0)
(1,0)
(2,0)



3.4 Why "0"?

Let $val : K^* \longrightarrow \mathbb{R}$ be given by $val(a) = min(q : \alpha_q \neq 0)$ and $f = \Sigma a x_1^{\alpha_1} ... x_n^{\alpha_n}$. Then,

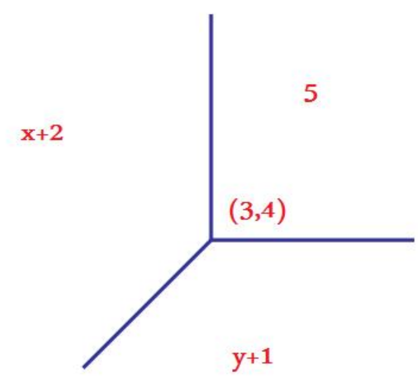
$trop(f) = min(val(a_1) + \alpha_1 x_1 + ... + \alpha_n x_n, val(a_2) + \alpha_1 x_1 + ... + \alpha_n x_n, ...)$

Example 16 *Let us consider polynomial $f(x, y) = t^2 x - 7(t + t^3)y + t^5$ where t is a variable in Laurent Polynomial ring $k[x_1^{\pm 1}, ... x_n^{\pm 1}]$. Like we defined $val(t)$ as the minimum exponent of the variable t, when we tropicalize polynomial $f(x, y)$ we get*

$$trop(f) = min(x + 2, y + 1, 5)$$

So we have three inequalities

(1) $x + 2 = y + 1 \leq 5 \implies x = y - 1$
(2) $x + 2 = 5 \leq y + 1 \implies x = -3$
(3) $y + 1 = 5 \leq x + 2 \implies y = 4$
The intersection of the lines $x = y - 1$, $x = -3$ and $y = 4$ is the point $(3, 4)$. Tropical hypersurface is



Example 17 *Let us see the tropical hypersurface defined by* $f(x,y) = tx^2+2xy+3ty^2+5x+7y-(t^2+t^5)$. *Tropicalization of this polynomial is*

$$trop(f) = min(2x+1, x+y, 2y+1, x, y, 2)$$

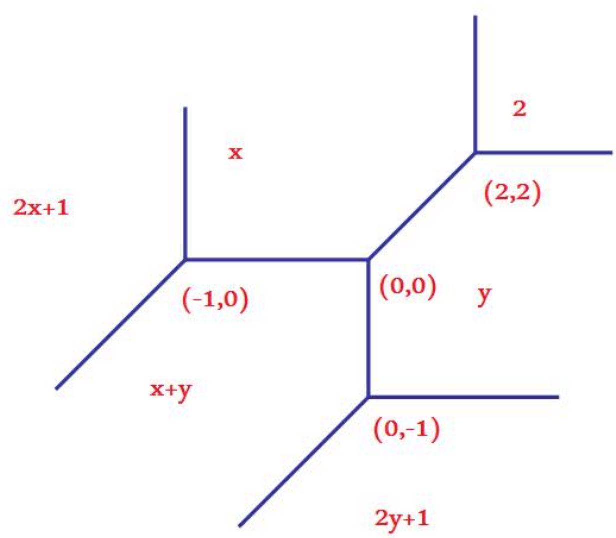
Considering tropical hypersurface of a tropical polynomial $f(x,y)$ *means, finding all the points* $x = (x,y)$ *where the maximum is attained by two or more of the linear functions. Therefore we will equate linear functions threesome.*

(1) $x = y = 2 \Longrightarrow (2,2)$
(2) $x = y = x+y \Longrightarrow (0,0)$
(3) $2x+1 = x = x+y \Longrightarrow (-1,0)$
(4) $2y+1 = y = x+y \Longrightarrow (0,-1)$

Then the set of all points (x,y) *where the function* $trop(f)$ *is piecewise-linear is as follows*

$$\{(2,2), (0,0), (-1,0), (0,-1)\}$$

Thus starting with this set we will clearly see at which part separated by the lines in (1), (2), (3) *and* (4) *the maximum is attained.*



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